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Impossibility of the antidynamo theorem for generic planar periodic flows

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The behavior of a nonzero-mean planar magnetic field in a conducting fluid within a periodic box is studied. It is shown that for a particular class of stationary flows either there is indefinite growth of the maximum of the magnetic field or the variation of this field within the box has as large an order as the diffusivity factor permits. Consequently the field cannot relax to any kind of uniform or near-uniform state in this kinematic setting. [S1063-651X(97)50906-8]

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There exists a number of results proving the impossibility of maintaining or increasing a certain type of magnetic field with a certain type of motion of a conducting fluid. Among the most celebrated of these so-called antidynamo theorems we have Cowling's theorem on the impossibility of maintaining axisymmetric fields with axisymmetric motions [1] (this result was much improved later) and the result of Zeldovich and Ruzmaikin asserting that a spatially periodic magnetic field of mean zero cannot be maintained by a steady or time-periodic planar periodic flow [2]. The zero-mean condition is essential in the proof. Now, it is also known that the so-called fast dynamos [3] work by a geometric process (stretch twist fold) which is impossible in planar flows; hence there is a general feeling that dynamo processes are unlikely in two dimensions with reasonable hypotheses upon the velocity. When disregarding the influence of the magnetic field upon the velocity, which is considered a datum (the kinematic dynamo problem), we are left with the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}(\mathbf{u}\times\mathbf{B} - \varepsilon\,\mathrm{curl}\mathbf{B}), \tag{1}$$

where $\mathbf{u}$ stands for the fluid velocity and ε for the fluid magnetic diffusivity. When both $\mathbf{u}$ and $\mathbf{B}$ are spatially periodic in a box $V=[0,\alpha]\times[0,\beta]\times[0,\gamma]$, the mean value

$$\langle \mathbf{B}\rangle = \int\int_V \mathbf{B} \tag{2}$$

is conserved in time. Thus there cannot be a general theorem stating that $\mathbf{B}$ tends to zero. Could there exist a result asserting that $\mathbf{B}$ tends to a uniform state? Indeed, if $\mathbf{u}$ is constant, $\mathbf{B}-\langle\mathbf{B}\rangle$ satisfies the induction equation and has mean zero, and so the Zeldovich theorem shows that it tends to zero. Although for a nonuniform velocity such a strong result is unlikely, one could expect the field to become more and more smooth in time by the effect of diffusion. That this is not so was shown by detailed analyses, to which we will refer later, of certain periodic problems by Childress and Soward [4–6]. We will show here that for a more general class of flows no smoothing of the field takes place: Either the field grows indefinitely in time or its spatial configuration becomes as wildly varying as permitted by the order of the diffusive term; i.e., $\|\nabla\mathbf{B}\|$ has an order of $1/\varepsilon$ (see [7] for a rigorous statement of this general assertion).

Let us now establish our hypotheses. We assume that both $\mathbf{u}=(u_x,u_y,0)$ and $\mathbf{B}=(B_x,B_y,0)$ are planar and periodic in V, $\mathbf{u}=\mathbf{u}(x,y)$, $\mathbf{B}=\mathbf{B}(x,y,t)$. Also $\mathrm{div}\mathbf{B}=0$ and $\mathrm{div}\mathbf{u}=0$. There exists a scalar vector potential $\phi(x,y,t)$ such that $\mathbf{B}=(-\partial\phi/\partial y,\partial\phi/\partial x,0)$. However, ϕ does not need to be periodic. Then the induction equation may be ''uncurled'' to show that ϕ satisfies

$$\frac{\partial\phi}{\partial t}\mathbf{e}_z = (\varepsilon\Delta\phi - \mathbf{u}\cdot\nabla\phi)\mathbf{e}_z + \nabla W, \tag{3}$$

where W is an unknown potential. Since ϕ does not depend on z, necessarily W has the form $zf(t)+g(t)$, and so we are left with

$$\frac{\partial\phi}{\partial t} = (\varepsilon\Delta\phi - \mathbf{u}\cdot\nabla\phi) + f(t). \tag{4}$$

Let us integrate this equation in V. Since $\nabla\phi$ is periodic, the integral of $\Delta\phi$ is zero. On the other hand,

$$\begin{aligned}\int\int_V \mathbf{u}\cdot\nabla\phi &= \int\int_V \mathrm{div}(\phi\mathbf{u}) = \int_{\partial V}\phi\mathbf{u}\cdot\mathbf{n}\\ &= \gamma\int_0^\alpha[\phi(x,\beta)-\phi(x,0)]u_y(x,0)dx\\ &\quad+\gamma\int_0^\beta[\phi(\alpha,y)-\phi(0,y)]u_x(0,y)dy \end{aligned}\tag{5}$$

(recall that u_x and u_y are periodic). Let us state our first hypothesis: Assume that u_y is constant in the boundary $y=0$ and u_x in $x=0$. We get

$$\begin{aligned}\int\int_V \mathbf{u}\cdot\nabla\phi &= \gamma u_y\int_0^\alpha\int_0^\beta\frac{\partial\phi}{\partial y}(x,y)dxdy\\ &\quad+\gamma u_x\int_0^\beta\int_0^\alpha\frac{\partial\phi}{\partial x}(x,y)dxdy\\ &= \gamma u_y\langle -B_x\rangle + \gamma u_x\langle B_y\rangle = C\alpha\beta\gamma. \end{aligned}\tag{6}$$

C is constant in time and independent of the election of ϕ. Obviously with a judicious election of the initial conditions of **B** and the boundary values of **u** we may obtain $C\neq 0$, for which we will need $\mathbf{u}\cdot\mathbf{n}\not\equiv 0$, $\langle \mathbf{B}\rangle \neq 0$. Let us take now another scalar potential for **B**,

$$\psi = \phi - \langle \phi(0)\rangle - \int_0^t f(s)ds + Ct. \tag{7}$$

Since $(\partial/\partial t)\langle\psi\rangle = 0$ and $\langle\psi(0)\rangle = 0$, ψ is the (unique) potential for **B** with mean zero.

Now we pose our second hypothesis: There exists a streamline not crossing the box boundaries. This could be a periodic streamline, a stagnation point, or have infinite length. If we prefer to consider that we are working in a torus instead of a periodic box, the streamline should be contractible to a point. Writing the above equation as

$$\frac{d\psi}{dt} = \varepsilon\Delta\psi + C, \tag{8}$$

where d/dt is the Lagrangian derivative, it is clear that we may obtain ψ by integrating $\varepsilon\Delta\psi + C$ along the streamline. Unless this integral is a bounded function, the maximum $\|\psi\|_\infty$ becomes arbitrarily large in time, although it does not need to grow monotonically. Since the mean of ψ in V is 0, at some point of V ψ vanishes, so that also $\|\mathbf{B}\|_\infty = \|\nabla\psi\|_\infty$ becomes arbitrarily large. We take this as an (perhaps intermitent) dynamo effect. The other alternative is that $\varepsilon\Delta\psi$ becomes arbitrarily close to C arbitrarily often in time. In this case $\|\Delta\psi\|_\infty$ must be of order C/ε for a sequence of times tending to ∞, and thus $\|\nabla\mathbf{B}\|_\infty$ has at least an order C/ε. This means that the gradient of **B** must be large in order to allow diffusion to dissipate the secular growth of ψ along the streamline produced by convection.

As stated above, particular cases of steady periodic motions, which share some characteristics with our model, have been studied in depth by Childress and Soward [4–6]. There are closed streamlines (cat's eyes) near the center of the cells and hyperbolic stagnation points near the boundaries. These authors show that the effects of the magnetic boundary layers created by the stagnation points plus the flux transported through the channels outside the cat's eyes may combine to create fields of order $\varepsilon^{-1/2}$ varying in a length scale of order $\varepsilon^{1/2}$, so that the order of $\|\nabla\mathbf{B}\|_\infty$ is $1/\varepsilon$, in accordance with our results.

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